Toric geometry, exercise sheet 6

Exercise 1. Let $M = \mathbb{Z}^3$ and consider the Reeve tetrahedron

$$R_2 = \mathrm{Conv}\{(0,0,0),(1,0,0),(0,1,0),(1,1,2)\} \subset M_{\mathbb{R}} = \mathbb{R}^3.$$

Show that $X_{R_2 \cap M}$ is smooth, while X_{R_2} is singular.

Exercise 2. Let $N = \mathbb{Z}^2$. Let Σ be the fan in $N_{\mathbb{R}}$ with the two maximal cones

$$\sigma_1 = \mathrm{Cone}\{(1,0),(1,1)\}, \qquad \sigma_2 = \mathrm{Cone}\{(1,1),(0,1)\}.$$

(a) Show that $X_\Sigma \cong \mathrm{Bl}_0 \mathbb{A}^2$.

(b) Complete Σ by adding as few smooth maximal cones as possible, and show that the resulting toric variety is the blow-up of $\mathbb{P}^2$ at a torus-fixed point, $\mathrm{Bl}_p \mathbb{P}^2$.

(c) Show that $\mathrm{Bl}_p \mathbb{P}^2$ can be embedded into $\mathbb{P}^4$ and give its defining equations.

Remark. $\mathrm{Bl}_p \mathbb{P}^2 \subset \mathbb{P}^4$ is not a complete intersection. Note that $\mathrm{Bl}_p \mathbb{P}^2 \subset \mathbb{P}^2 \times \mathbb{P}^1 \subset \mathbb{P}^5$ via the Segre embedding; we found an embedding into a smaller projective space.

Exercise 3.

(a) Find 16 pairwise non-equivalent lattice polygons $P_1, \ldots, P_{16} \subset M_{\mathbb{R}} = \mathbb{R}^2$ with exactly one interior lattice point, up to translations and $\mathrm{GL}_2(\mathbb{Z})$.

(b) Show that exactly 5 of them are smooth, say $P_1, \ldots, P_5$.

(c) Show that the normal fans Σ_{P_i} for $i = 1, \ldots, 5$ are $\mathrm{GL}_2(\mathbb{Z})$-equivalent to the fan of $\mathbb{P}^1 \times \mathbb{P}^1$ or to a refinement of the fan of $\mathbb{P}^2$ obtained by adding at most three rays.

Remark. The five smooth polygons correspond to the five smooth toric del Pezzo surfaces: $\mathbb{P}^1 \times \mathbb{P}^1$ and the blow-ups of $\mathbb{P}^2$ at $k = 0, 1, 2, 3$ torus-fixed points (with successive blow-ups when needed). The remaining eleven polygons yield singular Gorenstein toric del Pezzo surfaces, which admit smoothings to smooth del Pezzo surfaces of degrees 3 through 8. Such a smoothing can also be toric; the smooth del Pezzo surfaces in these degrees are $\mathbb{P}^1 \times \mathbb{P}^1$ (degree 8) and blow-ups of $\mathbb{P}^2$ at $1 \leq k \leq 6$ points in general position. Their anticanonical bundles are very ample.

Exercise 4. Prove the following proposition.

Proposition. Let $P \subset M_{\mathbb{R}}$ be a full-dimensional lattice polytope. Then its normal fan Σ_P is a complete fan.