Toric geometry, exercise sheet 5

Exercise 1. For an integer $n \geq 0$ consider the lattice polytope

$$P_n = \mathrm{Conv}\big\{(0,0),(n+1,0),(0,1),(1,1)\big\} \subset \mathbb{R}^2.$$

The projective toric variety $X_{P\cap A}$ is a *Hirzebruch surface* (with a suitable projective embedding).

(a) Compute the Ehrhart polynomial $E_P(k) = |(k \cdot P) \cap \mathbb{Z}^2|$ and $\deg(X_{P\cap A})$.

(b) Show that P is smooth, i.e., for every vertex $v \in V(P)$, the cone $\sigma_v := \mathrm{Cone}(P - v)^\vee \subset N_\mathbb{R}$ is smooth.

(c) Show that P is normal.

Remark. Oda's conjecture states that every smooth lattice polytope is normal.

Exercise 2. Let e_1, e_2, e_3 be the standard basis of $M = \mathbb{Z}^3$. For $n \geq 1$, define the *Reeve tetrahedron*

$$R_n = \mathrm{Conv}\{0, e_1, e_2, e_1 + e_2 + ne_3\} \subset \mathbb{R}^3.$$

Recall that the *Minkowski sum* of subsets $A, B \subset \mathbb{R}^3$ is $A + B = \{a + b \mid a \in A,\ b \in B\}$.

(a) Show that R_n is not very ample for $n > 1$.

(b) Show that $2 \cdot R_n$ is very ample for all $n \geq 1$.

Remark. In fact, $2 \cdot R_n$ is normal.

(c) Show that $P = R_4 + [0, e_3]$ is very ample but not normal.

Remark. Hence, the converse of Proposition 3.39 does not hold.

Remark. Every lattice polygon is normal, since it admits a unimodular triangulation. The example above is the easiest example of a lattice polytope which is very ample but not normal.

Exercise 3. Prove the following proposition:

Proposition. If a full-dimensional lattice polytope $P \subset M_\mathbb{R}$ is normal, then $X_{P\cap M}$ is projectively normal in the embedding defined by $P \cap M$.