

TORIC GEOMETRY, EXERCISE SHEET 3

Exercise 1. Let $N = \mathbb{Z}^2$, $M = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$, and

$$\sigma = \text{Cone}\left((1, 0), (2, 5)\right) \subseteq N_{\mathbb{R}}.$$

- (1) Compute the dual cone $\sigma^{\vee} \subseteq M_{\mathbb{R}}$ and the Hilbert basis of $S_{\sigma} = \sigma^{\vee} \cap M$.
- (2) Determine generators and defining equations for the coordinate ring $\mathbb{C}[S_{\sigma}]$. In particular, realize U_{σ} as a subvariety of affine space.
Hint: You may use a result by Riemenschneider: If the Hilbert basis of S_{σ} consists of n elements, then it defines a toric variety $U_{\sigma} \subseteq \mathbb{C}^n$ whose ideal has $(n-1)(n-2)/2$ generators.

Exercise 2. Let $S \subseteq \mathbb{Z}^2$ be the affine semigroup generated by

$$(2, 0), \quad (0, 1), \quad (1, 1),$$

and let $Y = \text{Spec } \mathbb{C}[S]$.

- (1) Compute $\mathbb{Z}S$, $\text{Cone}(S)$, and all lattice points in $\text{Cone}(S) \cap \mathbb{Z}S$ that are not contained in S .
- (2) Show that $Y \subseteq \mathbb{C}^3$ is a hypersurface and find the defining equation of Y .
- (3) Determine the normalization $\tilde{Y}$ and give the normalization morphism $\nu : \tilde{Y} \rightarrow Y$ explicitly in coordinates.
- (4) Determine the singular locus of Y . Where is ν an isomorphism, and what are the fibers over the remaining points?

Exercise 3. Prove

Theorem 2.55. Let V be an affine toric variety with dense torus T , character lattice M , and cocharacter lattice $N = \text{Hom}(M, \mathbb{Z})$. The following are equivalent:

- (i) V is normal.
- (ii) $V \cong \text{Spec } \mathbb{C}[S]$ for a finitely generated semigroup $S \subseteq M$ which is saturated in M .
- (iii) $V \cong U_{\sigma} = \text{Spec } \mathbb{C}[\sigma^{\vee} \cap M]$ for some strongly convex rational polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$.

Exercise 4. Prove

Proposition. Let S be an affine semigroup, put $M = \mathbb{Z}S$ and $N = \text{Hom}(M, \mathbb{Z})$, and set

$$C = \text{Cone}(S) \subseteq M_{\mathbb{R}}, \quad \sigma = C^{\vee} \subseteq N_{\mathbb{R}}.$$

Then σ is a strongly convex rational polyhedral cone, and the inclusion

$$\mathbb{C}[S] \hookrightarrow \mathbb{C}[\sigma^{\vee} \cap M]$$

induces the normalization morphism

$$U_{\sigma} \longrightarrow V = \text{Spec } \mathbb{C}[S].$$