

TORIC GEOMETRY, EXERCISE SHEET 2

Exercise 1. For the affine toric varieties in Example 2.9, find a lattice N and a cone $\sigma \subseteq N_{\mathbb{R}}$ with U_{σ} isomorphic to the given variety, or explain why no such cone exists.
Hint: We will see that U_{σ} is always a normal variety.

- (a) The torus $(\mathbb{C}^*)^n$.
- (b) Affine space $\mathbb{C}^n$.
- (c) The cuspidal cubic $V(y^2 - x^3) \subseteq \mathbb{C}^2$.
- (d) The threefold node $V(xy - zw) \subseteq \mathbb{C}^4$.

Exercise 2. Let $\sigma_i \subseteq (N_i)_{\mathbb{R}}$ be rational convex polyhedral cones for $i = 1, 2$. Their product is the cone

$$\sigma_1 \times \sigma_2 = \{(v_1, v_2) \in (N_1 \oplus N_2)_{\mathbb{R}} : v_i \in \sigma_i\}.$$

Prove that

$$U_{\sigma_1 \times \sigma_2} \cong U_{\sigma_1} \times U_{\sigma_2}.$$

Exercise 3. For $n = 1, 2$, classify all rational convex polyhedral cones σ in $\mathbb{R}^n$ up to $\mathrm{GL}_n(\mathbb{Z})$ -equivalence. Compute their dual cones $\sigma^{\vee}$ and say as much as you can about the affine toric variety U_{σ} .

Exercise 4. Prove

Proposition 2.33. Let $\sigma \subseteq N_{\mathbb{R}}$ be a cone. Then

- (1) $\sigma^{\vee} \subseteq M_{\mathbb{R}}$ is a rational convex polyhedral cone.
- (2) $(\sigma^{\vee})^{\vee} = \sigma$ (identifying $N_{\mathbb{R}}$ with its double dual).
- (3) For $m_1, \dots, m_s \in M$,

$$\sigma^{\vee} = \mathrm{Cone}(m_1, \dots, m_s) \iff \sigma = H_{m_1}^+ \cap \dots \cap H_{m_s}^+.$$

Exercise 5. Prove

Proposition 2.34. Let $\sigma \subseteq N_{\mathbb{R}}$ be a cone. Then

- (0) If $\tau \preceq \sigma$, then τ is a rational convex polyhedral cone.
- (1) If $\tau, \tau' \preceq \sigma$, then $\tau \cap \tau' \preceq \sigma$.
- (2) If $\tau \preceq \sigma$ and $\tau' \preceq \tau$, then $\tau' \preceq \sigma$.
- (3) If $\tau \preceq \sigma$ and $v, w \in \sigma$, then $v + w \in \tau$ implies $v, w \in \tau$.
- (4) There is an inclusion-reversing bijection

$$\{\text{faces of } \sigma\} \longleftrightarrow \{\text{faces of } \sigma^{\vee}\}.$$

Describe the correspondence explicitly.