

Exercise 1. For each of the following affine toric varieties from Example 2.9, find a suitable lattice M and finite subset $\mathcal{A} \subseteq M$ such that $Y_{\mathcal{A}}$ is the given variety. Write down $\Phi_{\mathcal{A}}$ and justify your answer.

- (a) The torus $T = (\mathbb{C}^*)^n$.
- (b) Affine space $\mathbb{C}^n$.
- (c) The cuspidal cubic $V(y^2 - x^3) \subseteq \mathbb{C}^2$.
- (d) The threefold node $V(xy - zw) \subseteq \mathbb{C}^4$.

Exercise 2. Let $d \geq 2$ and choose $\mathcal{A} = \{(0, 1), (1, 1), \dots, (d, 1)\} \subseteq \mathbb{Z}^2$.

- (a) Show that $I(Y_{\mathcal{A}})$ is generated by the 2×2 minors of

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_{d-1} \\ x_1 & x_2 & \cdots & x_d \end{pmatrix}.$$

In other words, prove

$$I(Y_{\mathcal{A}}) = (x_i x_{j+1} - x_{i+1} x_j \mid 0 \leq i < j \leq d-1).$$

- (b) Consider the alternative set

$$\mathcal{B} = \{(d, 0), (d-1, 1), \dots, (0, d)\} \subseteq \mathbb{Z}^2.$$

Determine $\mathbb{Z}\mathcal{A}$ and $\mathbb{Z}\mathcal{B}$. Find an injective lattice homomorphism $L : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$ with $L(\mathcal{A}) = \mathcal{B}$, and calculate the index of its image.

- (c) Show that $Y_{\mathcal{A}} = Y_{\mathcal{B}}$ by comparing the monomial maps.
- (d) Prove that $Y_{\mathcal{A}}$ is also the image of

$$\psi : \mathbb{C}^2 \longrightarrow \mathbb{C}^{d+1}, \quad (u, v) \longmapsto (u^d, u^{d-1}v, \dots, uv^{d-1}, v^d).$$

Conclude that $Y_{\mathcal{A}}$ is the affine cone over the degree- d rational normal curve

$$C_d = \{[u^d : u^{d-1}v : \dots : v^d] \mid [u : v] \in \mathbb{P}^1\} \subseteq \mathbb{P}^d.$$

Exercise 3. Prove the following lemmas.

Lemma 2.13. Let $\varphi : T_1 \rightarrow T_2$ be a morphism of tori. The image $\varphi(T_1)$ is a closed subtorus of T_2 . *Hint: Use the Smith normal form for matrices with entries in a principal ideal domain (like $\mathbb{Z}$).*

Lemma 2.11. Let $\varphi : T_1 \rightarrow T_2$ be a morphism of tori. Then there is an induced group homomorphism of character lattices

$$\widehat{\varphi} : M(T_2) \longrightarrow M(T_1), \quad \chi^{\widehat{\varphi}(m)} = \chi^m \circ \varphi.$$

Moreover,

$$\begin{aligned} \varphi \text{ injective} &\iff \widehat{\varphi} \text{ surjective,} \\ \varphi \text{ surjective} &\iff \widehat{\varphi} \text{ injective.} \end{aligned}$$

Lemma 2.12. Let T be a torus with character lattice M . Let $\mathcal{A} = \{m_1, \dots, m_s\} \subseteq M$. Lemma 2.11 applied to the morphism of tori $\Phi_{\mathcal{A}} : T \rightarrow (\mathbb{C}^*)^s$ induces the group homomorphism of character lattices given by

$$\widehat{\Phi}_{\mathcal{A}} : \mathbb{Z}^s \longrightarrow M, \quad (a_1, \dots, a_s) \longmapsto \sum_{i=1}^s a_i m_i.$$